MAC 2311
Calculus I

Sections 4.3 & 4.5
Graphing Techniques

The Process

if you have a graphing calculator

1. Find $f(x) \to$ store in y_1
   $f'(x) \to$ store in y_2
   $f''(x) \to$ store in y_3

2. Original function analysis $\to$ Intercepts, Asymptotes

Intercepts

x-intercept $\to$ let $y=0$

y-intercept $\to$ let $x=0$ (table point)

Asymptotes

Vertical Asymptotes (VA) — set denominator $= 0$

Horizontal Asymptotes (HA) & other asymptotes

$\lim\limits_{x \to \infty} f(x)$ & $\lim\limits_{x \to -\infty} f(x)$

$\boxed{HA}$ continued

$\boxed{\text{College Algebra shortcut}}$ — be careful to make sure $\lim_{x\to\infty}$ & $\lim_{x\to-\infty}$ would not be different

$$f(x) = \frac{ax^n + \cdots}{bx^d + \cdots}$$

$\boxed{n<d}$ HA $y=0$

$\boxed{n=d}$ HA $y=\frac{a}{b}$

$\boxed{n>d}$ no HA but others exist
find it using long division.

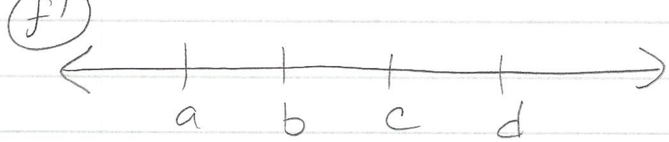
3 $\boxed{\text{First Derivative Analysis}}$ $\boxed{\begin{array}{c}\text{max / mins}\\ \text{incr / decr}\end{array}}$

set $f'(x)=0$ $f'(x)$ undefined

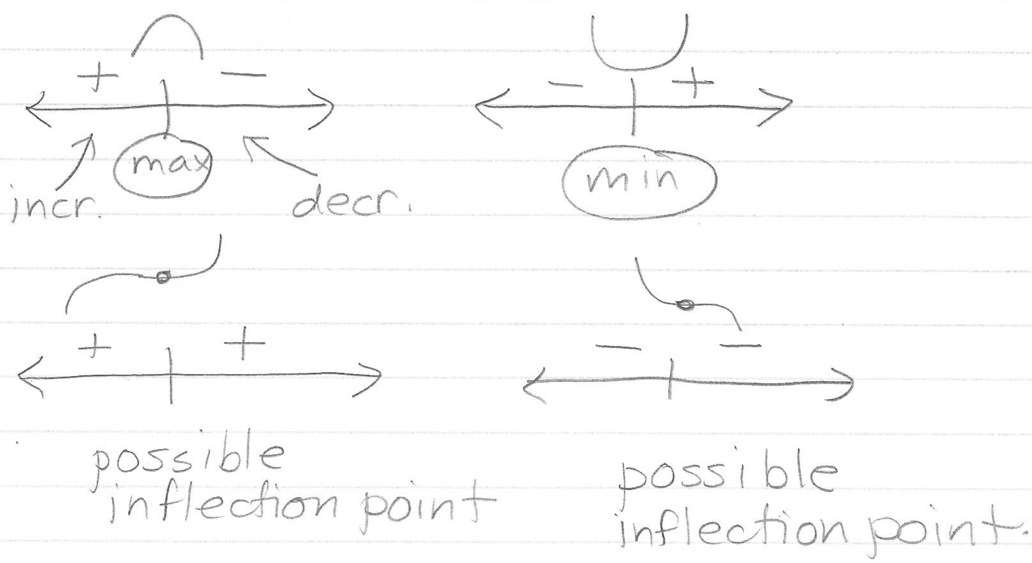
stationary points $\cap$ singular points $\wedge$

All values obtained are called critical numbers.
They are potential maximum & minimum values

Let a, b, c, & d be critical numbers



pick a test point in each area & analyze the results. You need to know the sign, not the number. Use f'



NOTE If a critical value is already determined to be a VA, it cannot be a max, min or inflection point.

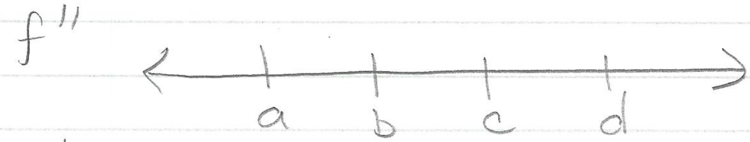
$f'(x) > 0$ increasing

$f'(x) < 0$ decreasing

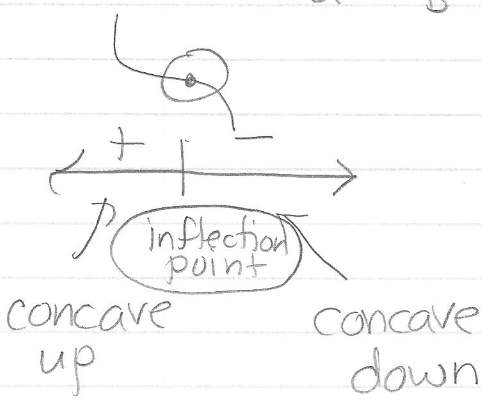
4 Second Derivative Analysis — concavity / inflection points

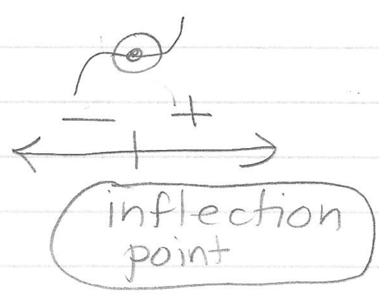
set $f''(x) = 0$ & f'' undefined

Let's say this generates values a, b, c & d

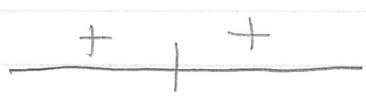


choose test points in each area and evaluate $f''(x)$

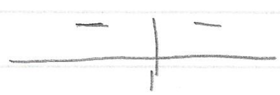




$f''(x) > 0$ concave up



not an inflection point



not an inflection point

$f''(x) < 0$ concave down

concave up
⇒ holds water
⇒ part of an upward parabola

concave down
⇒ spills water
⇒ part of a downward parabola

5 $\boxed{\text{Put it all together \& graph the functions}}$

Suggestions

1. Recopy all important information like the HA, VA, intercepts, y' number line, y'' number line in one location.

2. Set up an ordered pair table and find the locations (y values) using the original function.

x	y	
a		max
b		min
c		VA

etc

Example

Sketch a graph of $f(x) = \dfrac{x^3}{x-2}$

(#20, section 4.5)

$$f(x) = \frac{x^3}{x-2} \quad y_1$$

$$f'(x) = \frac{2x^2(x-3)}{(x-2)^2} \quad y_2$$

$$f''(x) = \frac{2x(x^2-6x+12)}{(x-2)^3} \quad y_3$$

Original function analysis

$$f(x) = \frac{x^3}{x-2}$$

x-int $y=0$

$$0 = \frac{x^3}{x-2} \Rightarrow 0 = x^3$$
$$x=0$$

so $\boxed{(0,0)}$

y-int $x=0$ $\boxed{(0,0)}$

$$f(0) = \frac{0^3}{0-2} = 0$$

$\boxed{\text{VA}}$ $f(x) = \dfrac{x^3}{x-2}$ $x-2=0$

$\boxed{x=2}$ VA

$\boxed{\text{HA}}$ we are analyzing $\dfrac{x^3}{x} \Rightarrow y = x^2$ format

$\boxed{\text{no HA}}$

$$\lim_{x\to\infty} \frac{x^3}{x-2} = \infty$$

$$\lim_{x\to -\infty} \frac{x^3}{x-2} = \infty$$

$\boxed{\text{other asymptote}}$ long division

$$\begin{array}{r|l} & x^2 + 2x + 4 + \dfrac{8}{x-2} \\ x-2 & x^3 + 0x^2 + 0x + 0 \\ & -(x^3 - 2x^2) \\ & \quad +2x^2 + 0x + 0 \\ & \quad -(+2x^2 - 4x) \\ & \qquad +4x + 0 \\ & \qquad -(4x - 8) \\ & \qquad\qquad 8 \end{array}$$

$y = x^2 + 2x + 4 = x^2 + 2x + 1 + 3$

$\boxed{y = (x+1)^2 + 3}$

parabola with vertex $(-1,3)$

First Derivative Analysis

$$f'(x) = \frac{2x^2(x-3)}{(x-2)^2}$$

$\boxed{f'(x) = 0}$

$2x^2(x-3) = 0$

$\boxed{x=0}$ $\boxed{x=3}$

$\boxed{f'(x) \text{ undef}}$

$x - 2 = 0$

$\boxed{x=2}$

VA

f'

	$(-\infty,0)$	0	$(0,2)$	2	$(2,3)$	3	$(3,\infty)$
sign	$-$		$-$		$-$		$+$
		IP ??		VA		min	

stored in y_2

test

$x = -1$ $\quad f'(-1) = \dfrac{2(-1)^2(-1-3)}{(-1-2)^2} = -$

$x = 1$ $\quad f'(1) = \dfrac{2(1)^2(1-3)}{(1-2)^2} = -$

$x = 2.5$ $\quad f'(2.5) = \dfrac{2(2.5)^2(2.5-3)}{(2.5-2)^2} = -$

$x = 4$ $\quad f'(4) = \dfrac{2(4)^2(4-3)}{(4-2)^2} = +$

increasing $(3, \infty)$

decreasing $(-\infty, 2)$ & $(2, 3)$

min when $x = 3$

possible IP when $x = 0$

Second Derivative Analysis

$$f''(x) = \frac{2x(x^2-6x+12)}{(x-2)^3}$$

$\boxed{f''(x)=0}$ (num)

$2x(x^2-6x+12)=0$

$2x=0$

$\boxed{x=0}$

~~$x^2-6x+12=0$~~

~~$x = \dfrac{+6 \pm \sqrt{36-4(1)(12)}}{2(1)}$~~

~~$x = \dfrac{6 \pm \sqrt{-12}}{2}$~~ ← not real

$\boxed{f''(x) \text{ undefined}}$ (denom)

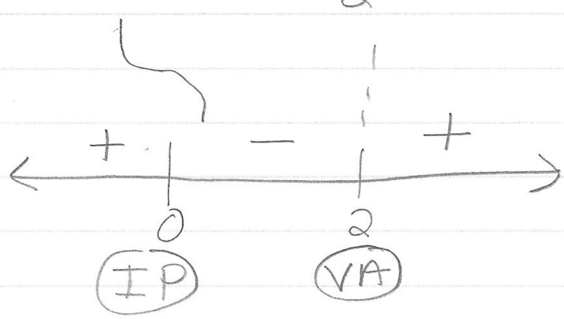
$x-2=0$

$\boxed{x=2}$

VA

f''

stored in y_3



$\boxed{\text{test}}$

$x=-1 \quad f''(-1) = \dfrac{2(-1)((-1)^2-6(-1)+12)}{(-1-2)^2}$ $+$

$x=1 \quad f''(1) = \dfrac{2(1)((1)^2-6(1)+12)}{(1-2)^2}$ $-$

$x=3 \quad f''(3) = \dfrac{2(3)(3^2-6(3)+12)}{(3-2)^2}$ $+$

$\boxed{\text{Summary}}$

concave up $(-\infty,0)$ & $(2,\infty)$

concave down $(0,2)$

Inflection point when $x=0$

The Graph

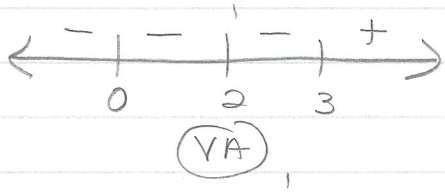
Important information gathered from all our groundwork:

Intercept (0,0)
min when $x = 3$
IP when $x = 0$

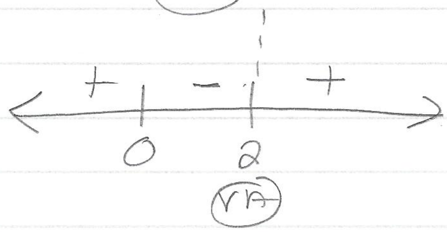
$\boxed{VA \quad x=2}$
$y = (x+1)^2 + 3$
parabolic asymptote

vertex
$(-1, 3)$
$(0, 4)$

f'

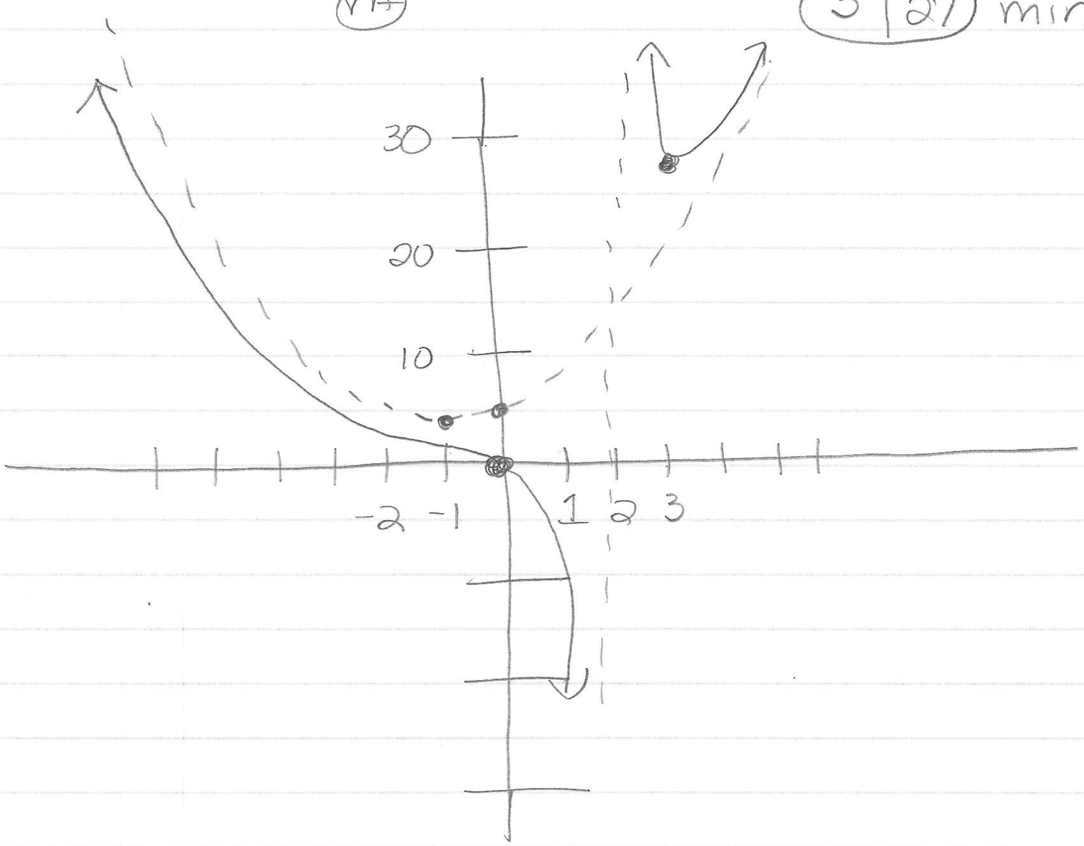


f''



x	f(x)	
0	0	
0	0	IP
3	27	min

points for the graph f(x)



Plot Extra points if you want to check.

x	f(x)
1	−1
−3	5.4